respectively. Equations (8) and (9) are identical with those for a two-dimensional boundary layer. It is evident that transformations (7) reduce to the corresponding expressions (1) for Newtonian fluids when $n = 1$.

It is well known that similar boundary layers exist for steady two-dimensional flows.[4] Since the equations for axisymmetric boundary layers reduce to those for two-dimensional boundary layers by the transformations (7), it is suggested that similar boundary layers exist also for steady axisymmetric flows. It will be shown in a separate paper[5] that this is true.

After the completion of this work, it was brought to the author's attention that Acrivos et al. have derived the same transformation in an unpublished paper.[6] Since, however, they confine their considerations to the cases where the velocity gradient in the boundary layer is always positive, the author believes that it would be worthwhile to present his own results.

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Equilibrium Equation for Radially Loaded Thin Circular Rings

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CONSIDER, in Fig. 1, an element ab of a thin circular ring of uniform cross section, symmetrical with respect to the plane of curvature, and subjected to distributed radial loading in this plane. Let R denote the radius of the centroidal axis of the ring, and $q(\theta)$ the intensity of radial load referred to this axis. On any cross section defined by θ, let N denote the normal force; Q, the shear force; and M, the bending moment; all considered positive in the directions shown.

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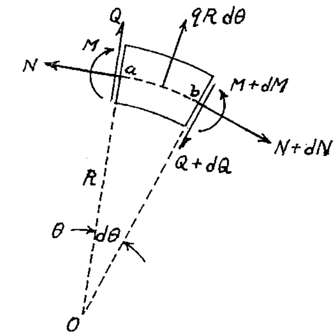

Fig. 1 Circular ring element with radical loading.

Summing forces in the radial direction $a0$ and neglecting small quantities of second order gives

$$qR\,d\theta - dQ - N\,d\theta = 0$$

from which

$$dQ/d\theta = qR - N \tag{1}$$

Summing moments about point 0 gives

$$dM = R\,dN \tag{2}$$

whereas summing moments about point b gives

$$dM/d\theta = QR \tag{3}$$

Elimination of Q and N from the equilibrium equations (1–3) gives now the third-order differential equation

$$(d^3M/d\theta^3) + (dM/d\theta) = R^2\,dq/d\theta \tag{4}$$

This expression can be integrated once to give

$$(d^2M/d\theta^2) + M = R^2\,q(\theta) + C \tag{5}$$

where C is a constant. If $q(\theta) = 0$, Eq. (5) gives $M =$ const. This simply means that any arbitrary uniform bending moment in the ring can be superimposed on that produced by the radial loading $q(\theta)$. Considering only bending caused by the q loading, we take $C = 0$ and write Eq. (5) in the form

$$(d^2M/d\theta^2) + M = R^2q(\theta) \tag{6}$$

Taking $s = R\theta$ as a new independent variable, Eq. (6) becomes

$$(d^2M/ds^2) + (M/R^2) = q(s) \tag{6a}$$

which has the same form as the familiar equation

$$(d^2w/ds^2) + (w/R^2) = M/EI \tag{7}$$

for radial deflection w of a circular ring. The existence of Eq. (6), analogous to Eq. (7) in form, does not seem to have been generally recognized. When $R = \infty$, both Eq. (6a) and Eq. (7) coincide with those for a straight bar.

Equation (6) can be useful in predicting buckling loads for uniformly compressed circular rings. When such a ring undergoes an inextensional bending during buckling, there will be a small change in curvature of the ring, as defined by the left-hand side of Eq. (7). As a result of this change in curvature, the existing normal forces N acting on each ele-

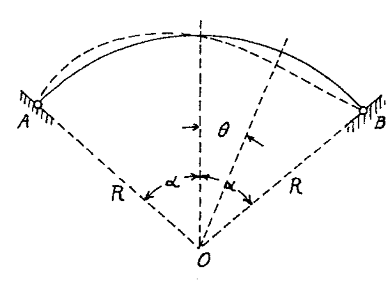

Fig. 2 Inextensional buckling of circular ring sector.

ment of the ring give rise to an additional radial loading of intensity

$$q_n = N[(d^2w/ds^2) + (w/R^2)] = NM/EI \tag{8}$$

where M is the bending moment associated with the change in curvature due to the buckling. Treating expression (8) as a fictitious radial loading, equilibrium of the ring in the buckled form can be studied on the basis of Eq. (6), which for this case becomes

$$(d^2M/d\theta^2) + M = -R^2MN/EI \tag{9}$$

N now being considered positive when it represents compression. Introducing the notation

$$n^2 = 1 + (R^2N/EI) \tag{10}$$

we can write Eq. (9) in the condensed form

$$d^2M/d\theta^2 + n^2M = 0 \tag{9a}$$

To illustrate the use of Eq. (9a) in studying the stability of a uniformly compressed circular ring, consider the sector hinged at its ends A and B and subtending a central angle 2α as shown in Fig. 2. In the circular solid line configuration, this ring is in equilibrium under the action of a uniform radial load of intensity $q = -p$ (not shown) and carries a corresponding uniform axial compression $N = pR$. To test whether this equilibrium configuration is stable or unstable, we superimpose an inextensional bending deformation, as shown by the dotted line AB, and use Eq. (9a). The general solution of this equation is

$$M = A\cos n\theta + B\sin n\theta \tag{11}$$

where A and B are constants. For the hinged ends of the sector, we have $M = 0$ when $\theta = \pm\alpha$. With these boundary conditions, Eq. (11) gives

$$\begin{aligned} 0 &= A\cos n\alpha + B\sin n\alpha \\ 0 &= A\cos n\alpha - B\sin n\alpha \end{aligned} \tag{12}$$

or, more simply,

$$A\cos n\alpha = 0 \qquad B\sin n\alpha = 0 \tag{13}$$

For inextensional buckling, we take $A = 0$ and $B \neq 0$. Then, for equilibrium in the buckled form, we must have

$$\sin n\alpha = 0$$

which yields the eigenvalues

$$n = m(\pi/\alpha) \tag{14}$$

where m is an integer. Returning to Eq. (10), we conclude this to mean that the critical value of the compressive force in the ring is

$$N_{cr} = [m^2(\pi^2/\alpha^2) - 1](EI/R^2) \tag{15a}$$

or, correspondingly, that the critical radial loading is

$$p_{cr} = [m^2(\pi^2/\alpha^2) - 1](EI/R^3) \tag{15b}$$

Taking the integer $m = 1$, we obtain the lowest critical values from (15a) and (15b).

In the case of a uniformly compressed full ring, we have $\alpha = \pi$ and Eqs. (15) become

$$N_{cr} = (m^2 - 1)(EI/R^2) \tag{16a}$$

$$p_{cr} = (m^2 - 1)(EI/R^3) \tag{16b}$$

In this case the integer $m = 1$ corresponds to a rigid body displacement of the ring, and the lowest buckling load is obtained by taking $m = 2$.

The results represented by Eqs. (15) and (16) are well known[1] but have always been obtained on the basis of Eq. (7). The purpose of this note is simply to call attention to the equilibrium equation (6) and its application in the treatment of curved bars. It is possible that a similar approach to other problems involving curved bars and shells might prove useful.

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Oblique Injection of a Jet into a Stream

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THE problem of a jet injected obliquely at an angle $\alpha(\pi > \alpha > 0)$ into a uniform stream can be used as a simple model to simulate many engineering problems, namely, the wing fan, the ground effect machine, etc. When the flow is assumed to be incompressible and inviscid, the jet will be bent by the main stream and will be bounded by two streamlines. They are the free streamline, which separates the jet from the wake behind it, and the interface, which separates the jet from the main stream (Fig. 1a). With the upstream uniform velocity U_∞ and the width of the jet b as the scales for velocity and length, respectively, the solutions of the problem will depend only on the dimensionless parameter $\varkappa$, which is defined as $(P_j - P)/(\frac{1}{2}\rho U_\infty^2)$, where P_j and P are the total pressure for the jet and the main stream, respectively. For example, the inclination of a streamline θ can be written, in general, as $\theta = \theta(x,y,\varkappa)$. Here, the pressure in the wake has been assumed to be equal to the pressure at upstream infinity. Otherwise, the solution would also depend on the cavitation number.

When the jet and the main stream have the same total pressure, i.e., $\varkappa = 0$, the velocity magnitude q and the inclination θ are continuous along the interface, and the potential solution for the jet and that for the main stream are therefore analytic continuations of each other. An analytic solution for both the jet and the main stream can be obtained by the standard technique for free streamline problems.[1] Figures 1b and 2a show, respectively, the complex potential χ plane, $\chi = \varphi + i\psi$, and the complex conjugate velocity w plane, $w = qe^{-i\theta}$. They are self-explanatory with the exception of the segment BC in the x plane which corresponds to the opening of the jet. We assume that the jet is louvered so that the flow direction may be specified, and thus the segment BC in the x plane obeys the condition

$$\left.\frac{d\psi}{d\varphi}\right|_{y=0} = \left(\frac{\partial\psi/\partial x}{\partial\varphi/\partial x}\right)_{y=0} = -\left(\frac{v}{u}\right) = -\tan\alpha \qquad 0 < x < 1$$

Hence, it is a straight line in the x plane inclined at an angle $-\alpha$. The standard technique is to map the circular sector in the w plane to the upper half plane and then to the open polygon in the x plane. Then the conformal mapping function $w(x)$ which can be obtained readily[2] can be used to relate x to z by the integral $z = \int dx/w$. The unknown constant ψ_j will then be related to the width of the jet b.

For the case $\varkappa \neq 0$, the pressure across the interface has to be continuous; therefore, there is a discontinuity in the magnitude of velocity. The discontinuity is defined by Ber-

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